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On the quadratic effect of random gravity waves on a vertical boundary

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The quadratic effect of random gravity waves in the vicinity of a reflecting boundary is studied. It is shown that in a stochastic wave environment, surface wave effects proportional to the square of the wave amplitude depend upon the third-order free-surface perturbation. Expressions are derived for the quadratic frequency spectrum of the hydrodynamic pressure in the fluid domain in unidirectional and standing waves reflected off a vertical wall. Computations of the spectrum reveal the importance of the effect contributed by the third-order solution, which is found to be at least of comparable magnitude to the corresponding effect obtained from the solution of the second-order problem.

1. Introduction

Nonlinear surface wave effects play an important role in the hydrodynamic analysis of vessels designed to operate in severe wave environments. They are for example known to excite low-frequency large-amplitude oscillations of compliant floating structures and are responsible for high-frequency loads experienced by offshore platforms. Numerous research efforts have therefore been devoted to the theoretical treatment of the nonlinear wave–body problem. In three dimensions, the majority of these studies are based on perturbation theory and have considered the solution of second-order interactions of surface waves with floating bodies. Early studies are reviewed by Ogilvie (1983) and more recent work by Faltinsen (1990).

This article studies the quadratic effects induced by random surface waves upon floating bodies, where 'quadratic' denotes all effects proportional to the square of the wave amplitude. They will be shown to be distinct from the 'second-order' effects considered in most studies to date. It is shown that the consistent account of all quadratic effects in a stochastic wave environment requires the solution of both the second-order and third-order wave–body problems.

In the absence of floating bodies, Tick (1959) derived and evaluated the second-order correction to the spectrum of unidirectional random waves. Tick did not account for third-order perturbations, although they contribute terms proportional to the square of the wave amplitude. Such effects were studied by Phillips (1960) in connection with the resonant interaction of surface wave components, while their relevance in the evaluation of the quadratic spectrum was pointed out by Hasselmann (1962) and Kinsman (1965). General expressions for the statistical distributions of surface-wave nonlinearities approximated by perturbation series of arbitrary order, were derived by Longuet-Higgins (1963).

The contribution of third-order perturbations to quadratic surface wave effects may be illustrated by studying an experimental record of the hydrodynamic pressure

$p(t)$ in the fluid domain, measured in a *deterministic* and in a *stochastic* wave field. The former may for example be generated by the monochromatic or bichromatic oscillation of a wavemaker, and $p(t)$ will typically be a periodic signal consisting of the primary and higher harmonics. The prediction of $p(t)$ by perturbation theory is obtained by the superposition of the linear solution $p_1(t)$, second-order solution $p_2(t)$ and if necessary the third-order solution $p_3(t)$. Each will contribute an effect of $O(A^n)$ respectively, where A is the wave amplitude and n the order of the perturbation. If effects of order up to A^2 are desired, it will be sufficient to consider the linear and second-order solutions.

In a stochastic wave environment, $p(t)$ as well as $p_i(t)$ will be random signals. Here the proper question to ask is, how many perturbation solutions $p_i(t)$ are necessary in order to approximate $p(t)$ to $O(A^2)$? An appropriate 'measure' of $p(t)$ of evident importance in practice, is its energy spectrum $\Phi(\omega)$ which may be readily measured from the record of $p(t)$. The 'linear' spectrum $\Phi_1(\omega)$ is of $O(A^2)$ and depends quadratically upon the linear solution $p_1(t)$. The 'quadratic' spectrum $\Phi_2(\omega)$ is of $O(A^4)$, and will consist of two components. The first depends on quadratic products of $p_2(t)$ while the second involves cross-products of $p_1(t)$ with $p_3(t)$. Therefore, the third-order solution will contribute to the quadratic spectrum an effect of the same order of magnitude as the second-order solution. This is the focal point of the present article which derives the complete form of the quadratic pressure spectrum in unidirectional and standing Gaussian waves, and presents computations which confirm the importance of the third-order solution.

In §2, the Pierson–Neumann theory for the description of a sea state is used for the spatial and temporal approximation of unidirectional and standing nonlinear random surface waves up to third order in the wave amplitude. Expressions are derived for the linear ($p_1(t)$), second-order ($p_2(t)$) and third-order ($p_2(t)$) perturbations of the hydrodynamic pressure in the fluid domain defined as stochastic processes and driven by an input stationary Gaussian sea state. Phillips (1960) has determined that resonant wave triads will arise in the third-order solution, shown to be responsible for the energy interchange between wave components in a sea state by Hasselmann (1962).

In §3 the complete form of the quadratic spectrum is derived, employing the expressions for $p_i(t)$, $i = 1, 2, 3$ obtained in §2. This analysis extends the derivation of Tick (1959) to third order, for primary wave components which propagate in the same or opposite directions. The resonance in the third-order solution gives rise to a Cauchy-type singularity in the definition of the quadratic spectrum, which must be interpreted in the principal value sense.

In §4, computations are presented of the quadratic spectrum in unidirectional and standing random waves for an input Pierson–Moskowitz spectrum for the free-surface elevation. In particular, the reflection of an input random wave disturbance off a vertical wall of infinite draught is considered in order to study the standing wave disturbance arising from the interaction of ambient waves with the body radiation and diffraction disturbances. Standing waves are known to give rise to a second-order pressure component which does not decay with depth and is considered responsible for high-frequency loads experienced by tension-leg platforms. A similar effect is present in the third-order solution, contributing a quadratic load in random waves as the second-order effect.

In unidirectional waves, the two components of the quadratic spectrum arising from the second- and third-order solutions are of comparable magnitude and contribute small corrections to the input linear spectrum of the pressure. At low frequencies, the former is dominant, while near the peak of the linear spectrum the latter is more significant. At high frequencies both decay rapidly to zero.

In standing waves, the magnitude of each component of the quadratic spectrum increases significantly, while two distinct features emerge. The component which depends on the second-order solution displays a distinct second peak at about twice the modal frequency of the input spectrum, and at high frequencies both components of the quadratic spectrum decay substantially more slowly than the linear spectrum. It may therefore be concluded that in the presence of a reflecting vertical boundary, or more generally a floating body, quadratic wave effects will contain a significant amount of energy at high frequencies contributed in comparable amounts by the second-order and third-order solutions of the wave–body interaction problem.

2. The perturbation expansion of random gravity waves

2.1. *The nonlinear problem*

Consider a nonlinear random gravity wave disturbance consisting of wave components propagating in the positive or negative x-directions, with the x-axis on the calm water surface. The free-surface elevation $\zeta(x,t)$ is assumed to be a homogeneous and stationary stochastic process, accepting the pair of Fourier–Stieljes representations

$$\zeta(x,t) = \int_k \int_\sigma A(k,\sigma)\, \mathrm{e}^{\mathrm{i}(kx-\sigma t)}\, \mathrm{d}k\, \mathrm{d}\sigma, \tag{2.1}$$

$$A(k,\sigma) = \frac{1}{(2\pi)^2} \int_x \int_t \zeta(x,t)\, \mathrm{e}^{-\mathrm{i}(kx-\sigma t)}\, \mathrm{d}x\, \mathrm{d}t. \tag{2.2}$$

Both are generalized Fourier integrals with the respective integrations extending to infinity (Lighthill 1959). Given a deep-water wave record $\zeta(x,t)$, the generalized Fourier transform $A(k,\sigma)$ is a random variable with zero mean, obeying the relation

$$\overline{A^*(k,\sigma)A(k-k',\sigma-\sigma')} = \Phi(k,\sigma)\,\delta(k')\,\delta(\sigma'). \tag{2.3}$$

The real function $\Phi(k,\sigma)$ is the frequency–wavenumber spectrum of this homogeneous and stationary wave field. Its properties and a more detailed discussion of relations (2.1)–(2.3) are given in Kinsman (1965).

The definition of the spectrum does not entail any knowledge of the underlying potential flow. The physics of this wave flow will lead to a relationship between k and σ and will allow representations analogous to (2.1) and (2.2) in the fluid domain.

Consider the potential $\phi(x,z,t)$ satisfying the two-dimensional Laplace equation in deep water and accepting the Fourier–Stieljes representation

$$\phi(x,z,t) = \int_k \int_\sigma B(k,\sigma)\, \mathrm{e}^{|k|z}\, \mathrm{e}^{\mathrm{i}(kx-\sigma t)}\, \mathrm{d}k\, \mathrm{d}\sigma. \tag{2.4}$$

A relation between $B(k,\sigma)$ and $A(k,\sigma)$ is obtained by enforcing the free-surface condition. The exact problem will not be pursued here. Instead the convergence of perturbation series expansion for the velocity potential and the wave elevation will be assumed,

$$\phi = \phi_1 + \phi_2 + \phi_3 \ldots, \tag{2.5}$$

$$\zeta = \zeta_1 + \zeta_2 + \zeta_3 + \ldots. \tag{2.6}$$

The boundary-value problems governing ϕ_i will be derived next and the Fourier–Stieljes representation of the wave field up to third order will be obtained.

2.2. *The linear problem*

The linear free-surface condition and wave elevation take the form

$$\frac{\partial^2 \phi_1}{\partial t^2}+g\frac{\partial \phi_1}{\partial z}=0, \quad z=0, \tag{2.7}$$

$$\zeta_1(x,t)=-\frac{1}{g}\frac{\partial \phi_1}{\partial t}. \tag{2.8}$$

Introducing a representation analogous to (2.4) for the linear potential ϕ_1 and enforcing the linear free-surface condition (2.7), we obtain

$$B_1(k,\sigma)(\sigma^2-g|k|)=0. \tag{2.9}$$

Solutions to (2.9) are of the form

$$B_1(k,\sigma)=B\left(\frac{\sigma|\sigma|}{g},\sigma\right)\delta\left(k-\frac{\sigma|\sigma|}{g},\sigma\right)+B\left(-\frac{\sigma|\sigma|}{g},\sigma\right)\sigma\left(k+\frac{\sigma|\sigma|}{g},\sigma\right), \tag{2.10}$$

where $\delta(x)$ is the Dirac delta function. The linear potential and corresponding wave elevation follow in the form

$$\phi_1^s(x,z,t)=\int_\sigma B_1^s(\sigma)\,\mathrm{e}^{|k|z}\exp\left[\mathrm{i}\sigma(s|\sigma|/g\,x-t)\right]\mathrm{d}\sigma, \tag{2.11}$$

$$\zeta_1^s(x,z,t)=\int_\sigma A_1^s(\sigma)\exp[\mathrm{i}\sigma(s\,|\sigma|/g\,x-t)]\,\mathrm{d}\sigma, \quad A_1^s(\sigma)=\frac{\mathrm{i}\sigma}{g}B_1^s(\sigma), \tag{2.12}$$

where $s=\pm 1$ for right- and left-going waves respectively, and $B_1^s=B_1(s\sigma|\sigma|/g)$. For a unidirectional wave field, $s=+1$ and it follows from (2.3) that

$$\overline{A_1^{+*}(\sigma)\,A_1^+(\sigma-\sigma')}=S_1(\sigma)\,\delta(\sigma'), \tag{2.13}$$

where $S_1(\sigma)$ is the two-sided frequency spectrum of the stationary and homogeneous linear wave field defined by (2.12).

2.3. *The second-order problem*

The ensuing derivation generalizes the analysis of Tick (1959) to second-order interactions between wavetrains propagating in the same or opposite directions.

For a linear wave field varying exponentially in the z-direction the second-order free-surface condition takes the form

$$\frac{\partial^2 \phi_2}{\partial t^2}+g\frac{\partial \phi_2}{\partial z}=-\frac{\partial}{\partial t}(\nabla\phi_1\cdot\nabla\phi_1) \tag{2.14}$$

enforced on the $z=0$ plane. The linear velocity potential here consists of wave components ϕ_1^s propagating in the positive or negative directions for $s=\pm 1$, therefore the second-order potential will accept the representation

$$\phi_2=\sum_{s_1 s_2}\phi_2^{s_1 s_2}, \tag{2.15}$$

where $s_1, s_2=\pm 1$ and

$$\phi_2^{s_1 s_2}(x,z,t)=\int_k\int_\sigma B_2^{s_1 s_2}(k,\sigma)\,\mathrm{e}^{|k|z}\,\mathrm{e}^{\mathrm{i}(kx-\sigma t)}\,\mathrm{d}k\,\mathrm{d}\sigma. \tag{2.16}$$

Substituting (2.11) and (2.16) in the free-surface condition (2.14) and invoking the coordinate transformation

$$k = (s_1 \sigma_1|\sigma_1| + s_2 \sigma_2|\sigma_2|)/g, \quad \sigma = \sigma_1+\sigma_2 \tag{2.17}$$

the second-order potential may be shown to accept the representation

$$\phi_2^{s_1 s_2}(x,z,t) = \int_{\sigma_1}\int_{\sigma_2} B_1^{s_1}(\sigma_1) B_1^{s_2}(\sigma_2)\, e^{|k_1+k_2|z+i[(k_1+k_2)x-(\sigma_1+\sigma_2)t]} \times \frac{Q^{s_1 s_2}(\sigma_1 \sigma_2)}{g|k_1+k_2|-(\sigma_1+\sigma_2)^2}\, d\sigma_1\, d\sigma_2, \tag{2.18}$$

where
$$Q^{s_1 s_2}(\sigma_1,\sigma_2) = i(\sigma_1+\sigma_2)(|k_1 k_2| - k_1 k_2), \tag{2.19}$$

with $k_i = s_i \sigma_i|\sigma_i|/g$. The double Fourier–Stieljes integral (2.18) is free of singularities. It follows from (2.19) that $Q^{s_1 s_2} = 0$ when $k_1 k_2 > 0$. When $k_1 k_2 < 0$, we may assume without loss of generality that $|\sigma_1| > |\sigma_2|$. It follows that

$$\frac{Q^{s_1 s_2}(\sigma_1,\sigma_2)}{g|k_1+k_2|-(\sigma_1+\sigma_2)^2} = -i\frac{\sigma_1^2\sigma_2}{g}. \tag{2.20}$$

The regularity of the ratio (2.20) suggests that the second-order free-surface condition (2.14) accepts no resonant solutions and leads to a second-order potential which is bounded in time.

2.4. *The third-order problem*

Carrying out the perturbation expansion of the nonlinear free-surface condition to third order, we obtain on the $z=0$ plane

$$\frac{\partial^2\phi_3}{\partial t^2}+g\frac{\partial\phi_3}{\partial z} = -2\frac{\partial}{\partial t}(\nabla\phi_1\cdot\nabla\phi_2)-\tfrac{1}{2}\nabla\phi_1\cdot\nabla(\nabla\phi_1\cdot\nabla\phi_1) -\zeta_1\frac{\partial}{\partial z}\left[\frac{\partial^2\phi_2}{\partial t^2}+g\frac{\partial\phi_2}{\partial z}+\frac{\partial}{\partial t}(\nabla\phi_1\cdot\nabla\phi_1)\right] = F_3(x,t), \tag{2.21}$$

where ζ_1 is given by (2.8). Solutions to (2.21) satisfying the Laplace equation in the fluid are obtained as in the second-order problem. Introducing the representations

$$F_3 = \sum_{s_1 s_2 s_3} F_3^{s_1 s_2 s_3}, \quad \phi_3 = \sum_{s_1 s_2 s_3} \phi_3^{s_1 s_2 s_3}, \tag{2.22}$$

where $s_i = \pm 1$ and using the linear and second-order solutions (2.11) and (2.18), it follows that

$$F_3^{s_1 s_2 s_3}(x,t) = \int_{\sigma_1}\int_{\sigma_2}\int_{\sigma_3} B_1^{s_1}(\sigma_1) B_1^{s_2}(\sigma_2) B_1^{s_3}(\sigma_3)\, e^{i[(k_1+k_2+k_3)x-(s_1+s_2+s_3)t]} \times C^{s_1 s_2 s_3}(\sigma_1,\sigma_2,\sigma_3)\, d\sigma_1\, d\sigma_2\, d\sigma_3, \tag{2.23}$$

with the cubic kernel $C^{s_1 s_2 s_3}$ defined as follows:

$$\begin{aligned} C^{s_1 s_2 s_3}(\sigma_1,\sigma_2,\sigma_3) = {} & \frac{-i\sigma_1}{g} Q^{s_2 s_3}(\sigma_2,\sigma_3)(|k_2+k_3|-k_2|-|k_3|) \\ & + [|k_1|\,|k_2+k_3| - k_1(k_2+k_3)]\frac{2i(\sigma_1+\sigma_2+\sigma_3)\, Q^{s_2 s_3}(\sigma_2,\sigma_3)}{g|k_2+k_3|-(\sigma_2+\sigma_3)^2} \\ & -\tfrac{1}{2}[|k_1|(|k_2|+|k_3|)-k_1(k_2+k_3)](|k_2 k_3| - k_2 k_3). \end{aligned} \tag{2.24}$$

Solutions to the Laplace equation subject to the linear free-surface condition and forced by the function (2.23), take the form

$$\phi_3^{s_1 s_2 s_3}(x,z,t) = \int_{\sigma_1}\int_{\sigma_2}\int_{\sigma_3} B_1^{s_1}(\sigma_1) B_1^{s_2}(\sigma_2) B_1^{s_3}(\sigma_3)\, e^{ikx-|k|z}\, \psi(k,\sigma;t) \times C^{s_1 s_2 s_3}(\sigma_1,\sigma_2,\sigma_3)\, d\sigma_1\, d\sigma_2\, d\sigma_3, \qquad (2.25)$$

where $k = k_1+k_2+k_3$, $\sigma = \sigma_1+\sigma_2+\sigma_3$ and the function $\psi(k,\sigma;t)$ obeys the differential equation

$$\frac{d^2\psi}{dt^2} + \sigma_0^2\psi = e^{-i\sigma t}, \qquad (2.26)$$

with $\sigma_0^2 = g|k|$. The particular solution of (2.26) is

$$\psi(\sigma,\sigma_0;t) = \begin{cases} \dfrac{e^{-i\sigma t}}{\sigma_0^2-\sigma^2}; & \sigma^2 \neq \sigma_0^2, \\ \dfrac{-t e^{-i\sigma t}}{2i\sigma}; & \sigma^2 = \sigma_0^2. \end{cases} \qquad (2.27)$$

It follows that when $\sigma_0^2 = \sigma^2$, or

$$g|k_1+k_2+k_3| = (\sigma_1+\sigma_2+\sigma_3)^2, \qquad (2.28)$$

the third-order velocity potential will grow in time owing to the resonant excitation of solutions to the homogeneous free-surface condition (2.30). Phillips (1960) has shown that this resonance is possible for appropriate combinations of the wavenumbers and frequencies of the primary wave components. Hasselmann (1962) carried out the perturbation expansion to fifth order and determined the resulting energy interchange among wave components in a random wave environment.

The perturbation series expansion of the hydrodynamic pressure $p(x,z,t)$ in the fluid domain follows from Bernoulli's equation and the perturbation potentials derived above. At $(x = 0,\ z = -d)$ the three leading terms of the pressure take the form:

linear pressure

$$p_1(0,-d,t) = \sum_{s_1}\int_{\sigma_1} A_1^{s_1}(\sigma_1) P_1(\sigma_1)\, e^{-i\sigma_1 t}\, d\sigma_1, \qquad (2.29)$$

$$P_1(\sigma_1) = \rho g\, e^{-|k_1|d}; \qquad (2.30)$$

second-order pressure

$$p_2(0,-d,t) = \sum_{s_1 s_2}\int_{\sigma_1}\int_{\sigma_2} A_1^{s_1}(\sigma_1) A_1^{s_2}(\sigma_2) P_2(\sigma_1,\sigma_2)\, e^{-i(\sigma_1+\sigma_2)t}\, d\sigma_1\, d\sigma_2, \qquad (2.31)$$

$$P_2(\sigma_1,\sigma_2) = \rho\frac{g^2}{\sigma_1\sigma_2}\left[\tfrac{1}{2}(|k_1 k_2| - k_1 k_2)\, e^{-(|k_1|+|k_2|)d} - i(\sigma_1+\sigma_2)\frac{Q^{s_1 s_2}(\sigma_1,\sigma_2)\, e^{-d|k_1+k_2|}}{g|k_1+k_2|-(\sigma_1+\sigma_2)^2}\right]; \qquad (2.32)$$

third-order pressure

$$p_3(0,-d,t) = \sum_{s_1 s_2 s_3}\int_{\sigma_1}\int_{\sigma_2}\int_{\sigma_3} A_1^{s_1}(\sigma_1) A_1^{s_2}(\sigma_2) A_1^{s_3}(\sigma_3) P_3(\sigma_1,\sigma_2,\sigma_3;t)\, d\sigma_1\, d\sigma_2\, d\sigma_3, \qquad (2.33)$$

$$P_3(\sigma_1,\sigma_2,\sigma_3;t) = \rho\frac{-\mathrm{i}g^3}{\sigma_1\sigma_2\sigma_3}\Bigg[C^{s_1s_2s_3}(\sigma_1,\sigma_2,\sigma_3)\,\mathrm{e}^{-d|k_1+k_2+k_3|}\frac{\mathrm{d}\psi(k,\sigma;t)}{\mathrm{d}t} + (|k_1|\,|k_2+k_3|-k_1(k_2+k_3))\frac{\mathrm{e}^{-(|k_1|+|k_2+k_3|)d}\,Q^{s_2s_3}(\sigma_2,\sigma_3)}{g|k_2+k_3|-(\sigma_2+\sigma_3)^2}\,\mathrm{e}^{-\mathrm{i}(\sigma_1+\sigma_2+\sigma_3)t}\Bigg]. \tag{2.34}$$

In the next section these expansions will be used to derive the quadratic component of the pressure spectrum.

3. The quadratic spectrum of the hydrodynamic pressure

In §2 the three leading terms in the perturbation series approximation of the hydrodynamic pressure at $x = 0$, $z = -d$ were derived. Each is a stochastic process, function of time with distinct statistical properties. We may therefore write

$$p(t) = p_1(t)+p_2(t)+p_3(t)+\dots. \tag{3.1}$$

First, the autocorrelation function of $p(t)$ is defined as the following ensemble average at some fixed time t:

$$R(\tau;t) = \mathrm{Re}\{E[p(t)\,p^*(t+\tau)]\}, \tag{3.2}$$

where Re stands for the real part and $*$ denotes the complex conjugate of the quantities involved. For stationary processes, the autocorrelation function is independent of t, yet here it is so far unknown if $p(t)$ is stationary owing to the form of the function $\psi(t)$ in the definition of the third-order perturbation $p_3(t)$.
The two-sided energy spectrum $\Phi(\omega)$ follows from the familiar relationship

$$\Phi(\omega;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\tau\,\mathrm{e}^{-\mathrm{i}\omega\tau}R(\tau;t). \tag{3.3}$$

The substitution of (3.1) into (3.2) leads to a series expansion for the autocorrelation function and the spectrum of the form

$$R(\tau;t) = R_1(\tau;t)+R_2(\tau;t)+\dots, \tag{3.4}$$

$$\Phi(\omega;t) = \Phi_1(\omega;t)+\Phi_2(\omega;t)+\dots, \tag{3.5}$$

where

$$R_1(\tau;t) = \mathrm{Re}\{E[p_1(t)\,p_1^*(t+\tau)]\} \tag{3.6}$$

$$R_2(\tau;t) = \mathrm{Re}\{E[p_2^*(t)\,p_2(t+\tau)+p_1^*(t)\,p_3(t+\tau)+p_1^*(t+\tau)\,p_3(t)]\}. \tag{3.7}$$

Our objective is the determination of the quadratic spectrum $\Phi_2(\omega)$ which may be seen from (3.7) to depend on all three leading-order perturbations of the hydrodynamic pressure.

For the evaluation of the ensemble averages indicated above, use will be made of (2.13), expressed in the more general form

$$E[A_1^{s_1}(\sigma_1)\,A_1^{s_2*}(\sigma_2)] = S^{s_1s_2}(\sigma_1)\,\delta(\sigma_1-\sigma_2). \tag{3.8}$$

Also, since the linear wave elevation defined by (2.12) is real, the following relation holds:

$$A_1^{s_1}(\sigma) = A_1^{s_1*}(-\sigma). \tag{3.9}$$

Equation (3.8) states that the generalized Fourier transforms of the linearized wave elevation defined by (2.12) are uncorrelated complex random variables at two unequal frequencies $\sigma_{1,2}$. When the associated wave fields are unidirectional, or $s_1 s_2 = 1$, the spectrum $S^{s_1 s_2}(\sigma)$ denotes the energy density of the linearized free-surface elevation at the frequency σ. When the input linear wave fields propagate in opposite directions, or $s_1 s_2 = -1$, the value of $S^{s_1 s_2}(\sigma)$ depends on their origin. If they arise from two independent storms, $A_1^+(s)$ is independent of $A_1^-(\sigma)$ and $S^{+-} = 0$. If on the other hand one wave field is the reflection of the other off a vertical wall, then their complex amplitudes of the same frequency are correlated and S^{+-} will be finite. These cases will be considered in more detail in §4 where unidirectional and standing wave fields are studied.

The validity of (3.8) does not entail that the linear free surface is Gaussian. The need for this assumption will become evident later in this section when the quadratic spectrum is derived.

In the linear problem, the substitution of the linear pressure defined by (2.29) in (3.6) leads to

$$R_1(\tau;t) = \operatorname{Re}\sum_{s_1 s_2}\int_{\sigma_1}\int_{\sigma_2} E[A_1^{s_1}(\sigma_1)A_1^{s_2 *}(\sigma_2)]\, P_1(\sigma_1)P_1^*(\sigma_2)\,\mathrm{e}^{-\mathrm{i}(\sigma_1-\sigma_2)t+\mathrm{i}\sigma_2\tau}\,\mathrm{d}\sigma_1\,\mathrm{d}\sigma_2, \tag{3.10}$$

and by virtue of (3.8), we obtain the linear autocorrelation function

$$R_1(\tau) = \sum_{s_1 s_2}\int_{-\infty}^{\infty} S^{s_1 s_2}(\sigma)|P_1(\sigma)|^2\,\mathrm{e}^{\mathrm{i}\sigma\tau}\,\mathrm{d}\sigma, \tag{3.11}$$

which may be verified to be independent of t because the linear pressure $p_1(t)$ is stationary. The linear spectrum follows from (3.3), in the form

$$\Phi_1(\omega) = |P_1(\omega)|^2 \sum_{s_1 s_2} S^{s_1 s_2}(\omega), \tag{3.12}$$

which is the familiar Wiener–Khintchine relation for linear systems.

The derivation of the quadratic spectrum follows similar steps. First decompose the corresponding autocorrelation function as follows:

$$R_2(\tau;t) = R_{22}(\tau;t)+R_{13}(\tau;t), \tag{3.13}$$

where

$$R_{22}(\tau;t) = \operatorname{Re}\{E[p_2(t)\,p_2^*(t+\tau)]\}, \tag{3.14}$$

$$R_{13}(\tau;t) = \operatorname{Re}\{E[p_1^*(t)\,p_3(t+\tau)+p_1^*(t+\tau)\,p_3(t)]\}. \tag{3.15}$$

For the first component, it follows upon substitution of (2.31) in (3.14) that

$$\begin{aligned} R_{22}(\tau;t) = \operatorname{Re}\sum_{s_1 s_2 s_3 s_4}\int_{\sigma_1}\int_{\sigma_2}\int_{\sigma_3}\int_{\sigma_4} & E[A_1^{s_1}(\sigma_1)A_1^{s_2}(\sigma_2)A_1^{s_3 *}(\sigma_3)A_1^{s_4 *}(\sigma_4)] \\ & \times P_2^{s_1 s_2}(\sigma_1,\sigma_2)\,P_2^{s_3 s_4}(\sigma_3,\sigma_4)\,\mathrm{e}^{-\mathrm{i}(\sigma_1+\sigma_2-\sigma_3-\sigma_4)t+\mathrm{i}(\sigma_3+\sigma_4)\tau}\,\mathrm{d}\sigma_1\,\mathrm{d}\sigma_2\,\mathrm{d}\sigma_3\,\mathrm{d}\sigma_4. \end{aligned} \tag{3.16}$$

Progress towards the further reduction of the fourfold integral (3.16) requires knowledge of the ensemble average of the product of four random variables. The property (3.8) is here of no direct use, unless further information on the statistical properties of the complex wave amplitudes $A_1^s(\sigma_i)$ is available. If the linear free-surface elevation defined by (2.12) is Gaussian, the uncorrelated random variables $x_i = A_1^s(\sigma_i)$ are jointly normally distributed and independent. In this case the following relation holds:

$$E[x_1 x_2 x_3 x_4] = E[x_1 x_2]E[x_3 x_4]+E[x_1 x_3]E[x_2 x_4]+E[x_1 x_4]E[x_2 x_3]. \tag{3.17}$$

For the random variables $A_1^{s_i}(\sigma_i)$, (3.17) and (3.8), (3.9) lead to the result

$$E[A_1^{s_1}(\sigma_1)A_1^{s_2}(\sigma_2)A_1^{s_3*}(\sigma_3)A_1^{s_4*}(\sigma_4)] = S^{s_1s_2}(\sigma_1)S^{s_3s_4}(\sigma_3)\delta(\sigma_1+\sigma_2)\delta(\sigma_3+\sigma_4)$$
$$+S^{s_1s_3}(\sigma_1)S^{s_2s_4}(\sigma_2)\delta(\sigma_1-\sigma_3)\delta(\sigma_2-\sigma_4)$$
$$+S^{s_1s_4}(\sigma_1)S^{s_2s_3}(\sigma_2)\delta(\sigma_1-\sigma_4)\delta(\sigma_2-\sigma_3). \quad (3.18)$$

The substitution of (3.18) in (3.16), reduces the autocorrelation function to the form

$$R_{22}(\tau) = \mathrm{Re}\sum_{s_1s_2s_3s_4}\iint_{-\infty}^{\infty} \mathrm{d}\sigma\,\mathrm{d}\sigma'\,[K_0^{s_1s_2s_3s_4}(\sigma,\sigma')+K^{s_1s_2s_3s_4}(\sigma,\sigma')\,\mathrm{e}^{\mathrm{i}(\sigma+\sigma')\tau}]. \quad (3.19)$$

where $$K_0^{s_1s_2s_3s_4}(\sigma,\sigma') = S^{s_1s_2}(\sigma)P_2^{s_1s_2}(\sigma,-\sigma')S^{s_3s_4}(\sigma')P_2^{s_3s_4}(\sigma,-\sigma'), \quad (3.20)$$

$$K^{s_1s_2s_3s_4}(\sigma,\sigma') = S^{s_1s_3}(\sigma)P_2^{s_1s_3}(\sigma,\sigma')S^{s_2s_4}(\sigma')P_2^{s_2s_4}(\sigma,\sigma')$$
$$+S^{s_1s_4}(\sigma)P_2^{s_1s_4}(\sigma,\sigma')S^{s_2s_3}(\sigma')P_2^{s_2s_3}(\sigma',\sigma). \quad (3.21)$$

The corresponding two-sided spectrum becomes

$$\Phi_{22}(\omega) = \sum_{s_1s_2s_3s_4}\left[\delta(\omega)\iint_{-\infty}^{\infty}\mathrm{d}\sigma\,\mathrm{d}\sigma'\,K_0^{s_1s_2s_3s_4}(\sigma,\sigma')+\int_{-\infty}^{\infty}\mathrm{d}\sigma\,K^{s_1s_2s_3s_4}(\sigma,\omega-\sigma)\right]. \quad (3.22)$$

The leading term in (3.22) arises from the non-zero mean value of the second-order pressure and will hereafter be omitted. The remaining term generalizes the corresponding expression derived by Tick (1959) for the wave elevation for wave components which may propagate in the same or opposite directions.

An equally important contribution to the quadratic spectrum arises from the autocorrelation function defined by (3.15). Using the definitions (2.29), (2.30) and (2.33), (2.34) of the linear and third-order pressures, we obtain

$$R_{13}(\tau;t) = \mathrm{Re}\sum_{s_1s_2s_3s_4}\int_{\sigma_1}\int_{\sigma_2}\int_{\sigma_3}\int_{\sigma_4} E[A_1^{s_1*}(\sigma_1)A_1^{s_2}(\sigma_2)A_1^{s_3}(\sigma_3)A_1^{s_4}(\sigma_4)]$$
$$\times P_1^{s_1*}(\sigma_1)[P_3^{s_2s_3s_4}(\sigma_2,\sigma_3,\sigma_4,t+\tau)\,\mathrm{e}^{\mathrm{i}\sigma_1 t}+P_3^{s_2s_3s_4}(\sigma_2,\sigma_3,\sigma_4,t)\,\mathrm{e}^{\mathrm{i}\sigma_1(t+\tau)}]\,\mathrm{d}\sigma_1\,\mathrm{d}\sigma_2\,\mathrm{d}\sigma_3\,\mathrm{d}\sigma_4. \quad (3.23)$$

Again, by virtue of (3.17) it follows that

$$E[A_1^{s_1*}(\sigma_1)A_1^{s_2}(\sigma_2)A_1^{s_3}(\sigma_3)A_1^{s_4}(\sigma_4)] = S^{s_1s_2}(\sigma_1)S^{s_3s_4}(\sigma_3)\delta(\sigma_1-\sigma_2)\delta(\sigma_3+\sigma_4)$$
$$+S^{s_1s_3}(\sigma_1)S^{s_2s_4}(\sigma_2)\delta(\sigma_1-\sigma_3)\delta(\sigma_2+\sigma_4)+S^{s_1s_4}(\sigma_1)S^{s_2s_3}(\sigma_2)\delta(\sigma_1-\sigma_4)\delta(\sigma_2+\sigma_3), \quad (3.24)$$

and upon substitution in (3.23) we obtain

$$R_{13}(\tau) = \sum_{s_1s_2s_3s_4}\iint_{-\infty}^{\infty}\mathrm{d}\sigma\,\mathrm{d}\sigma'\,P_1^{s_1}(\sigma)\,[S^{s_1s_2}(\sigma)S^{s_3s_4}(\sigma')L_1^{s_2s_3s_4}(\sigma,\sigma';t,\tau)$$
$$+S^{s_1s_3}(\sigma)S^{s_2s_4}(\sigma')L_2^{s_2s_3s_4}(\sigma,\sigma';t,\tau)+S^{s_1s_4}(\sigma)S^{s_2s_3}(\sigma')L_3^{s_2s_3s_4}(\sigma,\sigma';t,\tau)], \quad (3.25)$$

where

$$L_1(\sigma,\sigma';t,\tau) = \mathrm{Re}\,[P_3(\sigma,\sigma',-\sigma';t)\,\mathrm{e}^{\mathrm{i}\sigma(t+\tau)}+P_3(\sigma,\sigma',-\sigma';t+\tau)\,\mathrm{e}^{\mathrm{i}\sigma t}], \quad (3.26)$$
$$L_2(\sigma,\sigma';t,\tau) = \mathrm{Re}\,[P_3(\sigma',\sigma,-\sigma';t)\,\mathrm{e}^{\mathrm{i}\sigma(t+\tau)}+P_3(\sigma',\sigma,-\sigma';t+\tau)\,\mathrm{e}^{\mathrm{i}\sigma t}], \quad (3.27)$$
$$L_3(\sigma,\sigma';t,\tau) = \mathrm{Re}\,[P_3(\sigma',-\sigma',\sigma;t)\,\mathrm{e}^{\mathrm{i}\sigma(t+\tau)}+P_3(\sigma',-\sigma';t+\tau)\,\mathrm{e}^{\mathrm{i}\sigma t}], \quad (3.28)$$

and the superscripts $s_2s_3s_4$ have been omitted for brevity. The third-order pressure transfer function is defined by (2.34). Here we are interested to determine the real part of (3.26)–(3.28). It follows from (2.34) that the time dependence of P_3 is of the form

$$P_3(t) = \mathrm{i}A\frac{\mathrm{d}\psi(t)}{\mathrm{d}t}+B\,\mathrm{e}^{-\mathrm{i}\sigma t}, \quad (3.29)$$

where $\psi(t)$ is defined by (2.27) and A, B are real time-independent functions of the frequencies σ', σ.

When $\sigma^2 \neq \sigma_0^2$,

$$P_3(t)\,\mathrm{e}^{\mathrm{i}\sigma(t+\tau)} = \left(\frac{A}{\sigma_0^2-\sigma^2}+B\right)\mathrm{e}^{\mathrm{i}\sigma\tau} \tag{3.30}$$

and upon substitution in (3.26), it follows that

$$L_1(\sigma,\sigma';\tau) = 2\cos(\sigma\tau)\left(\frac{A}{\sigma_0^2-\sigma^2}+B\right). \tag{3.31}$$

For $\sigma^2 = \sigma_0^2$, the corresponding expressions become

$$P_3(t)\,\mathrm{e}^{\mathrm{i}\sigma(t+\tau)} = \left(-\mathrm{i}\frac{At}{2\sigma}+B\right)\mathrm{e}^{\mathrm{i}\sigma\tau} \tag{3.32}$$

and

$$L_1(\sigma,\sigma';\tau) = 2\cos(\sigma\tau)B. \tag{3.33}$$

It is evident from (3.33) that the linear time growth associated with the third-order pressure does not affect the evaluation of the quadratic wave spectrum because the corresponding third-order wave component is 90° out of phase relative to the linear solution. Thus, (3.26)–(3.28) reduce to

$$L_1(\sigma,\sigma';t,\tau) = 2P_3(\sigma,\sigma',-\sigma')\cos(\sigma\tau), \tag{3.34}$$

$$L_2(\sigma,\sigma';t,\tau) = 2P_3(\sigma',\sigma,-\sigma')\cos(\sigma\tau), \tag{3.35}$$

$$L_3(\sigma,\sigma';t,\tau) = 2P_3(\sigma',-\sigma',\sigma)\cos(\sigma\tau), \tag{3.36}$$

where the new third-order pressure transfer function P_3 is independent of time and is defined by

$$\begin{aligned} P_3(\sigma_1,\sigma_2,\sigma_3;t) = -\rho\frac{g^3}{\sigma_1\sigma_2\sigma_3}\Bigg[&\frac{(\sigma_1+\sigma_2+\sigma_3)\,C^{s_1s_2s_3}(\sigma_1,\sigma_2,\sigma_3)\,\mathrm{e}^{-d|k_1+k_2+k_3|}}{g|k_1+k_2+k_3|-(\sigma_1+\sigma_2+\sigma_3)^2} \\ &+\mathrm{i}(|k_1|\,|k_2+k_3|-k_1(k_2+k_3))\frac{\mathrm{e}^{-(|k_1|+|k_2+k_3|)d}\,Q^{s_2s_3}(\sigma_2,\sigma_3)}{g|k_2+k_3|-(\sigma_2+\sigma_3)^2}\Bigg], \end{aligned} \tag{3.37}$$

with $k_i = s_i\,\sigma_i|\sigma_i|/g$. Expression (3.37) is now a real quantity and the singularity at $g|k_1+k_2+k_3| = (\sigma_1+\sigma_2+\sigma_3)^2$ in the integral (3.25) must be interpreted in the Cauchy principal-value sense.

Upon substitution of (3.34)–(3.36) in (3.25), we obtain the autocorrelation function

$$R_{13}(\tau) = 2\sum_{s_1s_2s_3s_4}\iint_{-\infty}^{\infty}\mathrm{d}\sigma\,\mathrm{d}\sigma'\cos(\sigma\tau)\,L^{s_1s_2s_3s_4}(\sigma,\sigma'), \tag{3.38}$$

where

$$\begin{aligned} L^{s_1s_2s_3s_4}(\sigma,\sigma') = P_1^{s_1}(\sigma)\,[&S^{s_1s_2}(\sigma)\,S^{s_3s_4}(\sigma')\,P_3(\sigma,\sigma',-\sigma') \\ &+S^{s_1s_3}(\sigma)\,S^{s_2s_4}(\sigma')\,P_3(\sigma',\sigma,-\sigma')+S^{s_1s_4}(\sigma)\,S^{s_2s_3}(\sigma')\,P_3(\sigma',-\sigma',\sigma)], \end{aligned} \tag{3.39}$$

with P_1 defined by (2.30) and P_3 by (3.37). The frequency spectrum follows from (3.3),

$$\Phi_{13}(\omega) = \sum_{s_1s_2s_3s_4}\int_{-\infty}^{\infty}\mathrm{d}\sigma[L^{s_1s_2s_3s_4}(\omega,\sigma)+L^{s_1s_2s_3s_4}(-\omega,\sigma)]. \tag{3.40}$$

Equation (3.40) completes the derivation of the component of the quadratic spectrum which depends upon the linear and third-order solutions. It is of the same order of magnitude as the second-order spectrum (3.22) and represents the principal result of this paper.

In summary, the complete quadratic spectrum of the hydrodynamic pressure induced at some depth d by an input linear Gaussian random wave field consisting of wave components propagating in either direction, is given by

$$\Phi_2(\omega) = \sum_{s_1 s_2 s_3 s_4} \int_{-\infty}^{\infty} \mathrm{d}\sigma [K^{s_1 s_2 s_3 s_4}(\sigma, \omega-\sigma) + L^{s_1 s_2 s_3 s_4}(\omega, \sigma) + L^{s_1 s_2 s_3 s_4}(-\omega, \sigma)], \quad (3.41)$$

where the kernels K and L are defined by (3.21) and (3.40), respectively and the integral in (3.40) must be interpreted in the principal-value sense at the Cauchy-type singularity of the kernels L. The summations are over all possible combinations of $(s_1 s_2 s_3 s_4)$, with $s_i = \pm 1$.

The theory derived in this Section is applied in §4 to the evaluation of the pressure spectrum due to unidirectional and standing wave fields.

4. Spectra of unidirectional and standing waves

Two cases are studied in the present section. First, the quadratic pressure spectrum arising from a unidirectional random wave field will be computed, given the spectrum of the ambient wave elevation. Next, the spectrum of the hydrodynamic pressure on a vertical wall will be evaluated in order to illustrate the quadratic effect of random surface waves on floating bodies. In both cases the relative importance of the linear spectrum and the two components of the quadratic spectrum will be discussed.

4.1. *Unidirectional waves*

The frequency spectrum of the input Gaussian free-surface elevation is assumed to be of the form

$$S(\omega) = \frac{1}{2}\frac{A}{|\omega|^5}\mathrm{e}^{-B/\omega^4}, \quad (4.1)$$

where the constants A, B depend on the wind speed and fetch of the storm and the factor $\frac{1}{2}$ indicates that (4.1) is to be understood as a two-sided spectrum, an even function of ω. Its one-sided form equals twice (4.1) defined over the positive frequency axis.

For a unidirectional random wave field propagating in the positive x-direction, $s_i = 1$, $i = 1, 2, 3, 4$ in all expressions of §3, and the corresponding sums consist of one term. It follows that

$$S^{s_i s_j}(\omega) = S(\omega). \quad (4.2)$$

The linear spectrum of the pressure follows from (3.12),

$$\Phi(\omega) = |P_1(\omega)|^2 S(\omega), \quad (4.3)$$

with the linear transfer function P_1 given by (2.30). The two components of the quadratic spectrum follow from (3.22) and (3.40). The former reduces to

$$\Phi_{22}(\omega) = 2\int_{-\infty}^{\infty} \mathrm{d}\sigma S(\sigma) S(\omega-\sigma) P_2^2(\sigma, \omega-\sigma), \quad (4.4)$$

with the second-order transfer function P_2 given by (2.32) with $s_1 = s_2 = 1$. The second component of the quadratic spectrum is obtained from (3.39) and (3.40) in the form

$$\Phi_{13}(\omega) = P_1(\omega) S(\omega) \int_{-\infty}^{\infty} \mathrm{d}\sigma S(\sigma) [G(\omega, \sigma) + G(-\omega, \sigma)], \quad (4.5)$$

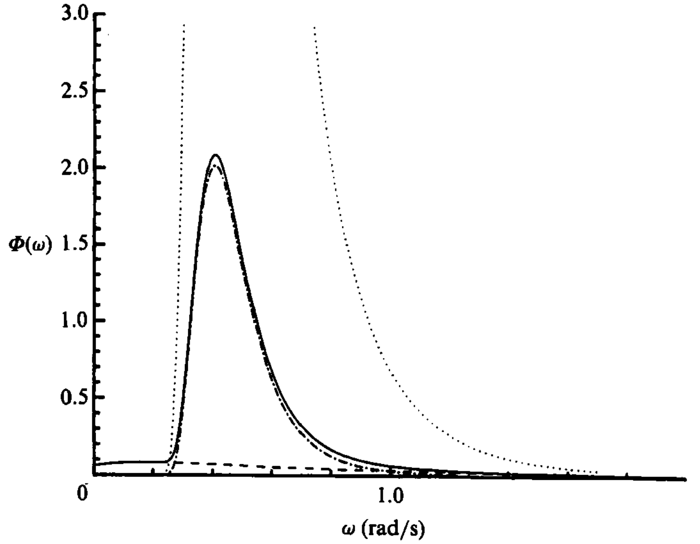

FIGURE 1. One-sided spectra of the hydrodynamic pressure $p/\rho g$ at a distance $d = 1$ m beneath the $z = 0$ plane in a unidirectional Gaussian sea state driven by a wind speed of 40 knots ..., linear spectrum with peak value at 16.7 m^2 s; —–, quadratic spectrum Φ_{22}; –·–, quadratic spectrum Φ_{13}; ——, total quadratic spectrum.

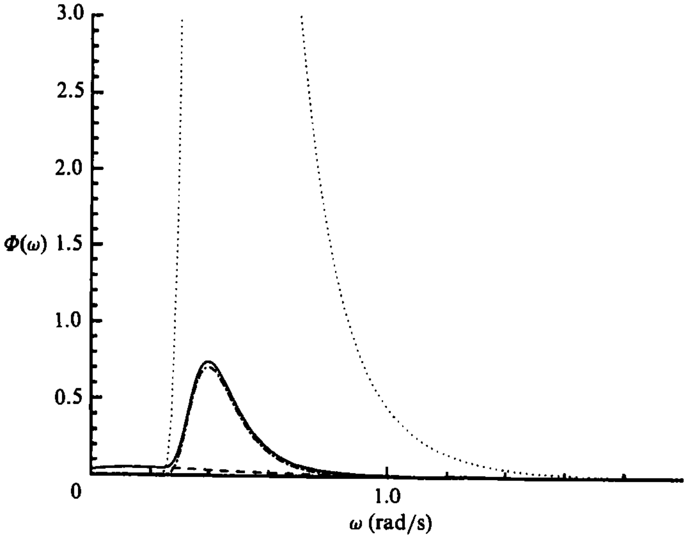

FIGURE 2. Same as figure 1 at water depth $d = 5$ m. Peak value of linear spectrum is at 15.9 m^2 s.

where
$$G(\omega, \sigma) = P_3(\omega, \sigma, -\sigma) + P_3(\sigma, \omega, -\sigma) + P_3(\sigma, -\sigma, \omega) \tag{4.6}$$
and the third-order pressure transfer function P_3 defined by (3.37). It follows from this definition with $s_i = 1$, that the resonance condition (2.37) is satisfied for all frequency triads which appear in (4.6), irrespective of the values of σ and ω. Therefore, it follows from (3.32), (3.33) that the first term in (3.37) is to be omitted, yielding a regular integrand in (4.5) for a unidirectional wave field.

Figures 1 and 2 illustrate the linear and quadratic frequency spectra of the hydrodynamic pressure at distances $d = 1$ and 5 m beneath the $z = 0$ plane. The input free-surface elevation is obtained from the Pierson–Moskowitz spectrum, (4.1), at a wind speed of 40 knots. The component $\Phi_{22}(\omega)$ of the quadratic spectrum may be seen to be small across the frequency range, but more pronounced at low frequencies.

The component Φ_{13} displays a significant peak at the modal frequency of the linear spectrum, which is however small relative to the corresponding linear value. This behaviour follows from (4.5), which indicates that Φ_{13} is proportional to the linear spectrum $S(\omega)$. A second consequence of this property is that Φ_{13} will be zero at low frequencies. Therefore, most of the quadratic energy density at low frequencies is contributed by the second-order component Φ_{22}. At high frequencies both components decay rapidly to zero.

4.2. *Standing waves*

Consider now the case of a random wave field propagating in the positive x-direction which encounters a vertical rigid wall of infinite draught located at $x = 0$. If the right-going linear wave field is defined by (2.11), (2.12) with $s = 1$, the reflected left-going field is given by the same relations with $s = -1$, and

$$A_1^-(\sigma) = A_1^+(\sigma). \tag{4.7}$$

It follows from (4.7) that right- and left-going wave components of the same frequency are correlated, and by virtue of (2.13) and (3.8) we obtain

$$S^{+-}(\sigma) = S(\sigma). \tag{4.8}$$

Therefore, all spectra in §3 of the form $S^{s_i s_j}(\sigma)$ are equal to the input spectrum $S(\sigma)$. The linear spectrum is obtained from (3.12) where the summation is over all four combinations of the index pair $s_1 s_2$. It follows that

$$\Phi(\omega) = 4|P_1(\omega)|^2 S(\omega). \tag{4.9}$$

The first component of the quadratic spectrum becomes

$$\Phi_{22}(\omega) = 2\int_{-\infty}^{\infty} \mathrm{d}\sigma\, S(\sigma)\, S(\sigma)\, S(\omega-\sigma)\left[\sum_{s_1 s_2} P_2^{s_1 s_2}(\sigma, \omega, -\sigma)\right]\left[\sum_{s_3 s_4} P_2^{s_3 s_4}(\sigma, \omega, -\sigma)\right], \tag{4.10}$$

where each summation in (4.10) is again over all four combinations of the index pairs $s_i\ s_j$. It follows from the definition (2.32) that the second-order transfer function obeys the symmetry relations $P_2^{s_1 s_2} = P_2^{s_2 s_1}$ which allows the reduction of (4.10) to the final form

$$\Phi_{22}(\omega) = 8\int_{-\infty}^{\infty} \mathrm{d}\sigma\, S(\sigma)\, S(\omega-\sigma)\left[\sum_{s=\pm 1} P_2^{s_1}(\sigma, \omega-\sigma)\right]^2. \tag{4.11}$$

Similar steps applied to the second component of the spectrum, lead to

$$\Phi_{13}(\omega) = 2P_1(\omega)\, S(\omega) \sum_{s_2 s_3 s_4} \int_{-\infty}^{\infty} \mathrm{d}\sigma\, S(\sigma)\, [G^{s_2 s_3 s_4}(\omega, \sigma) + G^{s_2 s_3 s_4}(-\omega, \sigma)], \tag{4.12}$$

with $G^{s_2 s_3 s_4}$ defined by (4.6) and (3.37) and the summation is over all eight combinations of the triad $s_2 s_3 s_4$ with $s_i = \pm 1$. Here, Cauchy-type singularities will arise in (4.12) because the indices s_i may assume positive and negative values. It follows from (2.37) that they will occur for values of σ such that

$$|2\sigma^2 \pm \omega^2| = \omega^2, \tag{4.13}$$

or at $\sigma = 0,\ \pm|\omega|$. All integrals in this section are evaluated by Romberg's

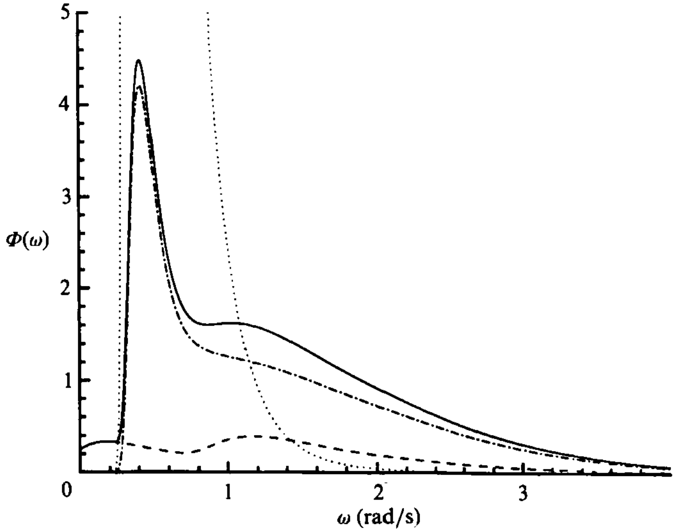

FIGURE 3. One-sided spectra of the hydrodynamic pressure $p/\rho g$ at a distance $d = 1$ m beneath the $z = 0$ plane in a standing Gaussian sea state driven by a wind speed of 40 knots: ..., linear spectrum with peak value at 67.3 m^2 s; ---, quadratic spectrum Φ_{22}; -·-, quadratic spectrum Φ_{13}; ——, total quadratic spectrum.

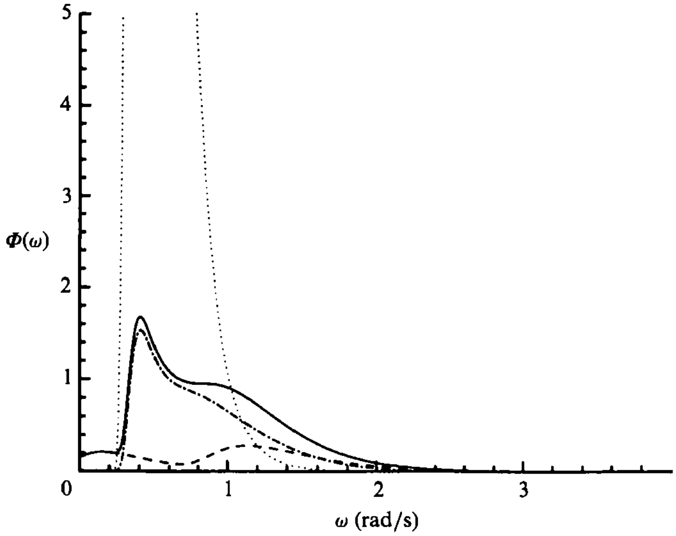

FIGURE 4. Same as figure 3 at a water depth $d = 5$ m. Peak value of linear spectrum is at 58.6 m^2 s.

quadrature. The integration in the vicinity of the Cauchy-type singularities is carried out by removing from the range of integration a segment centred at the singularity, and ensuring convergence as its length is allowed to approach zero.

Figures 3 and 4 illustrate the linear and quadratic components of the standing-wave pressure spectrum on a rigid wall located at $x = 0$ at depths $d = 1$ and 5 m respectively. The standing wave field is obtained by the reflection off the wall of an input unidirectional sea state propagating in the positive x-direction with energy density given by the Pierson–Moskowitz spectrum at a wind speed of 40 knots.

The magnitude of the quadratic components of the pressure spectrum are now significantly larger. In particular, the spectrum $\Phi_{22}(\omega)$ now displays a second peak at a frequency about twice the value of the modal frequency of the linear spectrum. It arises from the slow attenuation with depth of the second-order pressure signal in standing waves at a frequency equal to the sum of the frequencies of the primary wave components. The decay of the spectrum Φ_{13} also appears to be significantly slower beyond a certain frequency relative to the case of unidirectional waves. The same spectrum also displays a significant peak at the modal frequency of the linear spectrum, suggested by the form of expression (4.12). At low frequencies most of the energy density is supplied by Φ_{22} while Φ_{13} is essentially zero. At high frequencies both spectra decay to zero at a rate significantly slower than the linear spectrum.

Comparing figures 3 and 4 a trend is evident. The magnitude of the spectrum Φ_{22} appears to be relatively insensitive to the depth d, while the magnitude of Φ_{13} decreases with depth. Near the free surface, the contribution from Φ_{13} is greater except at low frequencies, while far from it its effect is reduced. At high frequencies the total quadratic pressure spectrum contains significantly more energy than would be predicted by linear theory. It is therefore concluded that both its components must be accounted for in the hydrodynamic analysis of marine structures like tension-leg platforms sensitive to high-frequency wave loads.

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